TROPICAL SPECTRAL CURVES AND INTEGRABLE CELLULAR AUTOMATA

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Abstract. We propose a method to study the integrable cellular automata with periodic boundary conditions, via the tropical spectral curve and its Jacobian. We introduce the tropical version of eigenvector map from the isolevel set to a divisor class on the tropical hyperelliptic curve. We also provide some conjectures related to the divisor class and the Jacobian. Finally, we apply our method to the periodic box and ball system and clarify the algebro-geometrical meaning of the real torus introduced for its initial value problem.

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1. Introduction

1.1. **Background and overview.** The box and ball system (BBS) [14] and the ultra-discrete Toda lattice [13] are typical examples of integrable cellular automata on one-dimensional lattice. The key to construct these systems from known soliton equations is a limiting procedure called *ultra-discretization* [15]. These automata are also well-defined on a periodic lattice, and those are what we study in this paper.

In [5], Kimijima and Tokihiro attempted to solve the initial value problem of the ultra-discrete periodic Toda lattice (UD-pToda). Their method consists of three steps: (1) send initial data of the UD-pToda to the discrete Toda lattice via inverse ultra-discretization, (2) solve the initial value problem for the discrete Toda lattice and (3) take the ultra-discrete limit. However, due to technical difficulties, this method has been completed only in the case of genus 1. Thereafter the initial value problem of the pBBS is solved by a combinatoric way [7] and by Bethe ansatz using Kerov-Kirillov-Reshetikhin bijection [6].

In this paper we propose a method to study the isolevel set of the UD-pToda and the pBBS via the tropical spectral curve and its Jacobian [9], intending to solve the initial value problem. We introduce the tropical version of eigenvector map from the isolevel set to a divisor class on the tropical hyperelliptic curve (Propositions 3.7, 3.10 etc). We provide some conjectures (Conjectures 2.3 and 3.4) related to the divisor class and the Jacobian, and also present concrete computation in the case of genus $g \leq 3$. Finally, by (4.5) we unveil the algebro-geometrical meaning of the real torus introduced in [6], on which the time evolution of the pBBS is linearized.

Key words and phrases. tropical geometry, integrable dynamical system, spectral curve, eigenvector map, Toda lattice.

Tropical geometry is being established recently by many authors (see [8, 12] and references therein for basic literature). It is defined over tropical semifield $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ equipped with the min-plus operation: "$x + y$" $= \min\{x, y\}$, "xy" $= x + y$. In [9], the Jacobian of a tropical curve has been introduced by means of the corresponding metric graph. Our approach might be a nice application of tropical geometry to integrable systems and one may confirm properness of the definition in [9].

1.2. **Tropical curve and UD-pToda.** We review on how tropical geometry appears in studying the UD-pToda lattice. Fix $g \in \mathbb{Z}_{>0}$. The $(g+1)$-periodic Toda lattice of discrete time $t \in \mathbb{Z}$ [4] is given by the difference equations on the phase space $\mathcal{U} = \{u^t = (I_1^t, \cdots, I_{g+1}^t, V_1^t, \cdots, V_{g+1}^t) \mid t \in \mathbb{Z}\} \simeq \mathbb{C}^{2(g+1)}$:

$$I_i^{t+1} = I_i^t + V_i^t - V_{i-1}^{t+1}, \qquad V_i^{t+1} = \frac{I_{i+1}^t V_i^t}{I_i^{t+1}}, \tag{1.1}$$

where we assume the periodicity $I_{i+g+1}^t = I_i^t$ and $V_{i+g+1}^t = V_i^t$. For each $u^t \in \mathcal{U}$, the Lax matrix is written as

$$L^t(y) = \begin{pmatrix} a_1^t & 1 & & & (-1)^g \frac{b_1^t}{y} \\ b_2^t & a_2^t & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & b_g^t & a_g^t & 1 \\ (-1)^g y & & & b_{g+1}^t & a_{g+1}^t \end{pmatrix}, \tag{1.2}$$

where $a_i^t = I_{i+1}^t + V_i^t$, $b_i^t = I_i^t V_i^t$ and $y \in \mathbb{C}$ is a spectral parameter. The evolution (1.1) preserves $\det(x\,\mathbb{I} + L^t(y))$. When we fix a polynomial $f(x, y) \in \mathbb{C}[x, y]$ as

$$f(x, y) = y^2 + y(x^{g+1} + c_g x^g + \cdots + c_1 x + c_0) + c_{-1}, \tag{1.3}$$

the isolevel set $\mathcal{U}_c$ for (1.1) is

$$\mathcal{U}_c = \{u^t \in \mathcal{U} \mid y \det(x\,\mathbb{I} + L^t(y)) = f(x, y)\}.$$

Let γ_c be the algebraic curve given by $f(x, y) = 0$. For generic c_i, γ_c is the hyperelliptic curve of genus g. Since the Lax matrix (1.2) is same as that for the original periodic Toda lattice (of continuous time) [1], $\mathcal{U}_c$ is isomorphic to the affine part of the Jacobi variety $\mathrm{Jac}(\gamma_c)$ of γ_c, and the time evolution (1.1) is linearized on $\mathrm{Jac}(\gamma_c)$ [5].

The *ultra-discrete limit* of (1.1) is the UD-pToda [11] given by the piecewise-linear map

$$T : \mathbb{R}^{2(g+1)} \to \mathbb{R}^{2(g+1)};\ (Q_i^t, W_j^t) \mapsto (Q_i^{t+1}, W_j^{t+1})$$

($t \in \mathbb{Z}$ and $i, j \in \{1, 2, \ldots, g+1\}$), where

$$Q_i^{t+1} = \min[W_i^t, Q_i^t - X_i^t], \qquad W_i^{t+1} = Q_{i+1}^t + W_i^t - Q_i^{t+1}, \tag{1.4}$$

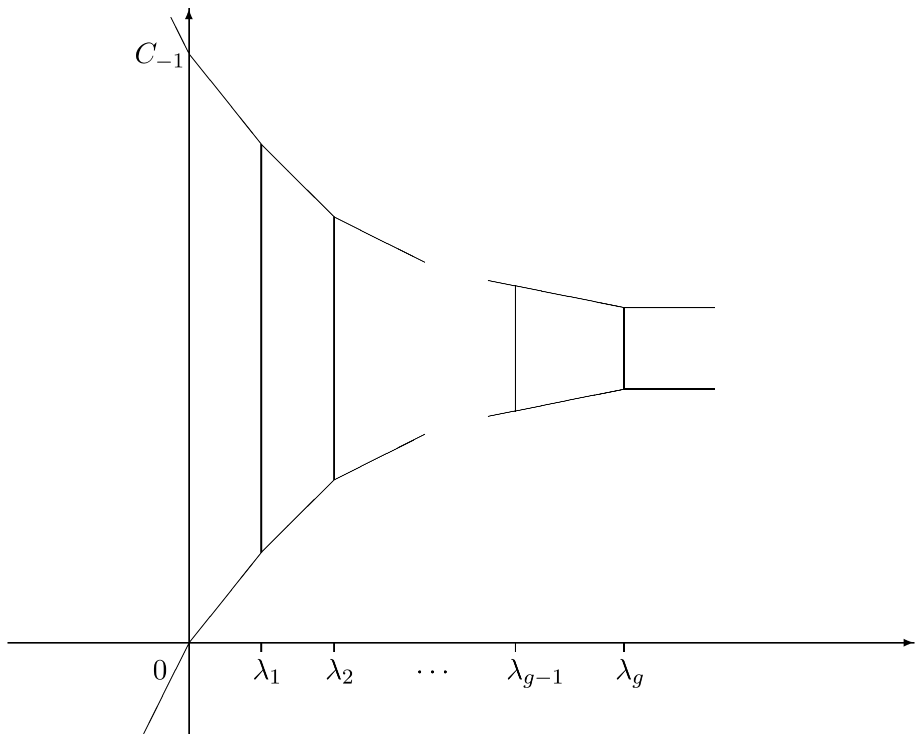

FIGURE 1. Tropical hyperelliptic curve

with $X_i^t = \min_{k=0,\cdots,g}\big[\sum_{l=1}^{k}(W_{i-l}^t - Q_{i-l}^t)\big]$. On the other hand, in this limit γ_c is reduced to the tropical curve $\tilde{\Gamma}_C \subset \mathbb{R}^2$ given by the polygonal lines of the convex in $\mathbb{R}^3$:

$$\{(X, Y, \min[2Y, (g+1)X + Y, gX + Y + C_g, \cdots, X + Y + C_1, Y + C_0, C_{-1}])\} \tag{1.5}$$

For generic C_i (see (2.1)), $\tilde{\Gamma}_C$ is *smooth* and depicted as Fig. 1 where we fix $C_g = 0$ and set $\lambda_i = C_{g-i} - C_{g-i+1}$ for $i = 1, \cdots, g$. Note that all edges of $\tilde{\Gamma}$ have fractional slopes.

We explicitly construct a tropical version of the eigenvector map from the isolevel set of the UD-pToda to the divisor class on $\tilde{\Gamma}_C$, and show that the isolevel set is isomorphic to the tropical Jacobi variety of $\tilde{\Gamma}_C$.

1.3. **Ultra-discrete limit and min-plus algebra.** We briefly introduce the notion of the ultra-discrete limit (UD-limit) and relate it to the min-plus algebra on the tropical semifield $\mathbb{T} = \mathbb{R} \cup \{\infty\}$.

We define a map $\mathrm{Log}_\varepsilon : \mathbb{R}_{>0} \to \mathbb{R}$ with an infinitesimal parameter $\varepsilon > 0$ by

$$\mathrm{Log}_\varepsilon : x \mapsto -\varepsilon \log x. \tag{1.6}$$

For $x > 0$, we define $X \in \mathbb{T}$ by $x = \mathrm{e}^{-\frac{X}{\varepsilon}}$. Then the limit $\varepsilon \to 0$ of $\mathrm{Log}_\varepsilon(x)$ converges to X. The procedure $\lim_{\varepsilon\to 0}\mathrm{Log}_\varepsilon$ with the scale transformation as $x = \mathrm{e}^{-\frac{X}{\varepsilon}}$ is called the ultra-discrete limit.

We summarize this procedure in more general setting:

Proposition 1.1. *For $A, B, C \in \mathbb{R}$ and $k_a, k_b, k_c > 0$, set*

$$a = k_a e^{-\frac{A}{\varepsilon}},\ b = k_b e^{-\frac{B}{\varepsilon}},\ c = k_c e^{-\frac{C}{\varepsilon}}$$

and take the limit $\varepsilon \to 0$ of the image Log_ε of the equations

$$\text{(i) } a + b = c,\ \text{(ii) } ab = c,\ \text{(iii) } a - b = c.$$

Then

$$\text{(i) } \min[A, B] = C,\ \text{(ii) } A + B = C$$

and

$$\text{(iii)} \begin{cases} A = C & (\text{if } \ A < B, \text{ or } A = B \text{ and } k_a > k_b) \\ \text{contradiction} & (\text{otherwise}) \end{cases}$$

hold.

1.4. **Content.** In §2, we define the metric graph Γ_C for the tropical hyperelliptic curve $\tilde{\Gamma}_C$ and define its Jacobian $J(\Gamma_C)$. By using a tropical version of the Abel-Jacobi map, we propose a divisor class which isomorphic to $J(\Gamma_C)$ at Conjecture 2.3. This is justified for $g \leq 3$. In §3, we study the isolevel set of the UD-pToda. We construct the eigenvector map from the isolevel set to the divisor class on the tropical curve. It is shown that the general level set is isomorphic to $J(\Gamma_C)$. In §4, we clarify the correspondence of the UD-pToda with the pBBS by refining that in [5]. In conclusion we interpret the isolevel set of the pBBS introduced in [6] in terms of tropical geometry.

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2. Tropical hyperelliptic curve and Jacobian

2.1. **Tropical hyperelliptic curve.** Fix $g \in \mathbb{Z}_{>0}$ and $C = (C_{-1}, C_0, \cdots, C_g) \in \mathbb{R}^{g+2}$. Let $\tilde{\Gamma}_C \subset \mathbb{R}^2$ be the affine tropical curve given by the polygonal lines of the convex in $\mathbb{R}^3$ (1.5). We assume a generic condition for C:

$$C_{-1} > 2C_0,\ C_i + C_{i+2} > 2C_{i+1}\ (i = 0, \cdots, g-2),\ C_{g-1} > 2C_g. \tag{2.1}$$

For simplicity, we fix $C_g = 0$ in the following. Define $\lambda = (\lambda_1, \cdots, \lambda_g)$ and $p_1, \cdots, p_g$ by

$$\lambda_i = C_{g-i} - C_{g-i+1}, \qquad p_i = C_{-1} - 2\sum_{j=1}^{g} \min[\lambda_i, \lambda_j]. \tag{2.2}$$

Under the condition (2.1) one sees $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_g$ and $2\sum_{i=1}^{g} \lambda_i < C_{-1}$.

By referring [8, Definition 2.18], we introduce a notion of smoothness of tropical curves:

Definition 2.1. The tropical curve $\Sigma \hookrightarrow \mathbb{R}^2$ is smooth if the following conditions are satisfied:
(a) all edges in Σ have fractional slopes.
(b) All vertex $v \in \Sigma$ is 3-valent.
(c) For each 3-valent vertex v, let e_1, e_2, e_3 be the oriented edges outgoing from v. Then the primitive tangent vectors ξ_k of e_k satisfy $\sum_{k=1}^{3} \xi_k = 0$, and $|\xi_k \wedge \xi_j| = 1$ for $k \neq j$, $k, j \in \{1, 2, 3\}$.

We see that $\tilde{\Gamma}_C$ is smooth. In particular, it is a tropical hyperelliptic curve whose genus is $\dim H_1(\tilde{\Gamma}_C, \mathbb{Z}) = g$ (see Fig. 1). We are to consider the maximal compact subset $\Gamma_C = \tilde{\Gamma}_C \setminus \{\text{infinite edges}\}$ of $\tilde{\Gamma}_C$. For simplicity we write Γ for Γ_C.

2.2. **Metric on Γ.** Following [9, §3.3], we equip Γ with the structure of a metric graph. Let $\mathcal{E}(\Gamma)$ be the set of edges in Γ, and define the weight $w : \mathcal{E}(\Gamma) \to \mathbb{R}_{\geq 0}$ by

$$w(e) = \frac{\| e \|}{\| \xi_e \|},$$

where ξ_e is the primitive tangent vector of $e \in \mathcal{E}(\Gamma)$, and $\| \ \|$ denotes any norm in $\mathbb{R}^2$. With this weight the tropical curve Γ becomes a metric graph.

The metric on Γ defines a symmetric bilinear form Q on the space of paths in Γ as follows: for a non-self-intersecting path γ, set $Q(\gamma, \gamma) := \text{length}_w(\gamma)$, and extending it to any pairs of paths bilinearly. In Fig. 2 we show the weight for each edge in Γ and the basis α_i $(i = 1, \cdots, g)$ of $\pi_1(\Gamma)$. For example, we have $Q(\alpha_1, \alpha_1) = C_{-1} + p_1 + 2\lambda_1$, $Q(\alpha_1, \alpha_2) = -p_1$, and $Q(\alpha_1, \alpha_i) = 0$ for $i > 2$.

2.3. **Tropical Jacobian.** Let $\Omega(\Gamma)$ be the space of global 1-forms on Γ, and $\Omega(\Gamma)^*$ be the dual space of $\Omega(\Gamma)$. Then both $\Omega(\Gamma)$ and $\Omega(\Gamma)^*$ are g dimensional and $\Omega(\Gamma)^*$ is isomorphic to $H_1(\Gamma, \mathbb{R})$.

Definition 2.2. [9, §6.1] The tropical Jacobian of Γ is a g dimensional real torus defined as

$$J(\Gamma) = \Omega(\Gamma)^* / H_1(\Gamma, \mathbb{Z}) \simeq \mathbb{R}^g \,/ K\, \mathbb{Z}^g \simeq \mathbb{R}^g \,/ \Lambda\, \mathbb{Z}^g \,.$$

Here $K, \Lambda \in M_g(\mathbb{R})$ are given by

$$\begin{aligned} K_{ij} =& Q(\alpha_i, \alpha_j), \\ \Lambda_{ij} =& Q(\sum_{k=1}^{i} \alpha_k, \sum_{l=1}^{j} \alpha_l) = C_{-1} + p_i \delta_{ij} + 2 \min[\lambda_i, \lambda_j]. \end{aligned}$$

Since Q is nondegenerate, K and Λ are symmetric and positive definite. In particular, we say that $J(\Gamma)$ is principally polarized.

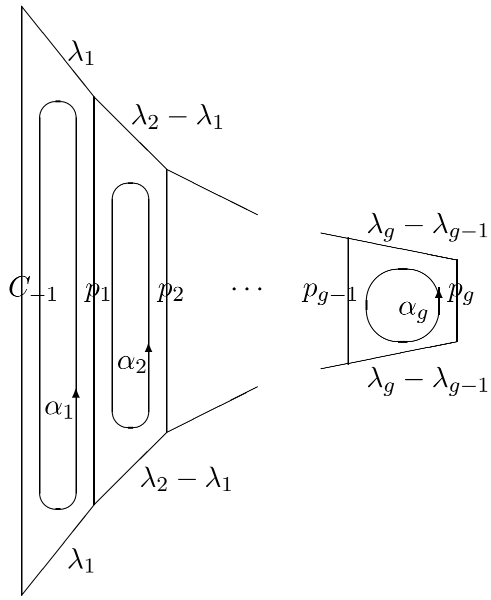

FIGURE 2. Γ_C as a metric graph

Let $\mathrm{Div}^g_{\mathrm{eff}}(\Gamma)$ be a set of effective divisors of degree g on Γ. We fix $P_0 \in \Gamma$ and define a map $\eta : \ \mathrm{Div}^g_{\mathrm{eff}}(\Gamma) \to J(\Gamma)$;

$$P_1 + \cdots + P_g \mapsto \sum_{i=1}^{g} (Q(\gamma_i, \alpha_1), \cdots, Q(\gamma_i, \alpha_g)), \tag{2.3}$$

where γ_i is the path from P_0 to P_i on Γ. Define $\alpha_{ij} = \alpha_i \cap \alpha_j \setminus \{\text{the end-points of } \alpha_i \cap \alpha_j\} \subset \Gamma$, and $\mathcal{D}^g(\Gamma)$ to be a subset of $\mathrm{Div}^g_{\mathrm{eff}}(\Gamma)$:

$$\mathcal{D}^g(\Gamma) = \left\{ P_1 + \cdots + P_g \;\middle|\; \begin{array}{l} P_i \in \alpha_i \text{ for all } i, \text{ and} \\ \text{there exists at most one point on } \alpha_{ij} \text{ for all } i \neq j \end{array} \right\}$$

Conjecture 2.3. *A reduced map $\eta|_{\mathcal{D}^g(\Gamma)}$ is bijective:*

$$\eta|_{\mathcal{D}^g(\Gamma)} : \ \mathcal{D}^g(\Gamma) \xrightarrow{\sim} J(\Gamma).$$

In the case of $g = 1$, this conjecture is obviously true since $\mathcal{D}^g(\Gamma) = \Gamma \simeq J(\Gamma)$ by definition. In the following we show that this conjecture is true for $g = 2$ and 3.

Proof. We define a map $\iota_S : \ \Gamma \to \mathbb{R}^g; \ P \mapsto \iota_S(P) = (Q(\gamma, \alpha_i))_{1 \le i \le g}$ where $S \in \Gamma$ and γ is an appropriate path from S to P. For $P_1 + \cdots + P_g \in \mathrm{Div}^g_{\mathrm{eff}}(\Gamma)$, we see $\eta(P_1 + \cdots + P_g) \sim \sum_{i=1}^{g} \iota_{P_0}(P_i)$ in $J(\Gamma)$.

$g = 2$ case: We set $P_0 = (\lambda_1, 2\lambda_1)$ which is the end-point of α_{12}. In the left figure of Fig. 3 we illustrate the locus of $\iota_{P_0}(P_i)$ where P_i starts from P_0 and moves along α_i for $i = 1, 2$ respectively. We set $O = (0, 0), A_1 = (C_{-1} + p_1 + 2\lambda_1, -p_1)$ and $A_2 = (-p_1, 2p_1)$. The parallelogram F of dash lines is the fundamental domain of $J(\Gamma)$. We calculate the

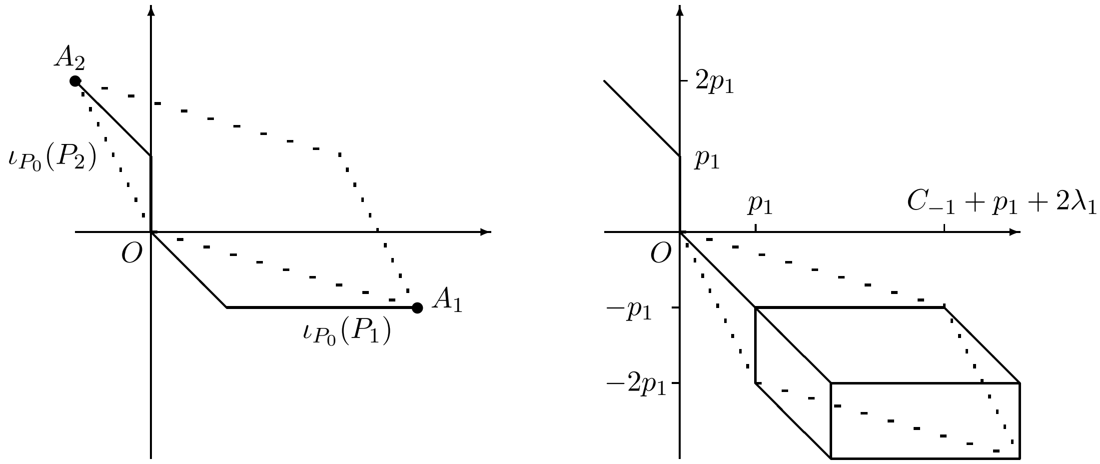

FIGURE 3. Imege of η in $g=2$

image of the map $\mathcal{D}^2(\Gamma) \to \mathbb{R}^2$ given by $P_1+P_2 \mapsto \iota_{P_0}(P_1)+\iota_{P_0}(P_2)+\overrightarrow{A_2O}$, and obtain the parallelohexagon V composed of three non-overlapped parallelograms as shown in the right figure of Fig. 3. It is easy to see that V is isomorphic to F in $J(\Gamma)$.

$g=3$ case: We set $S_1=(\lambda_1, 3\lambda_1)$ and $S_2=(\lambda_2, \lambda_1+2\lambda_2)$, the end-points of α_{12} and α_{23} respectively. We calculate the image of the map $\mathcal{D}^3(\Gamma) \to \mathbb{R}^3$ given by $P_1+P_2+P_3 \mapsto \iota_{S_1}(P_1)+\iota_{S_1}(P_2)+\iota_{S_2}(P_3)$, and obtain non-overlapped 12 parallelopipeds. After shifting some parallelopipeds along the lattice $K\,\mathbb{Z}^3$, we obtain the parallelo dodecahedron V in Fig. 4. We set $O=(0,0,0)$, $A_1=(C_{-1}+p_1+2\lambda_1, -p_1, 0)$, $A_2=(p_1, -p_1-p_2-2(\lambda_2-\lambda_2), p_2)$, $A_3=(0,-p_2,2p_2)$, $P=A_1+A_2+A_3$ and $B_i=A_j+A_k$ for $\{i,j,k\}=\{1,2,3\}$. The parallelopiped F spanned by $\overrightarrow{OA_1}, \overrightarrow{OA_2}$ and $\overrightarrow{OA_3}$ is the fundamental domain of $J(\Gamma)$. We draw V in black, and F in blue.

One sees that V coincides with F in $J(\Gamma)$ as follows: The polygon $V \setminus F$ is composed of three parts each of which contains the face $*'$ in z_2z_3-plane, the face $\star$ in z_1z_3-plane or the face $\diamond'$ in z_1z_2-plane. We translate the part with the face $*'$ (resp. $\star$, $\diamond'$) by $\overrightarrow{OA_1}$ (resp. $\overrightarrow{OA_2}$, $\overrightarrow{OA_3}$) and attach it on the face $*$ (resp. $\star'$, $\diamond$). $\square$

Remark 2.4. After this paper was submitted, we proved Conjecture 2.3 for general g in another way, by applying the notion of rational functions on Γ [3, 9].

For § 4.2, we introduce another torus $J'(\Gamma)$:

$$J'(\Gamma) = \mathbb{R}^g \,/A\,\mathbb{Z}^g, \tag{2.4}$$

where $A_{ij} = \Lambda_{ij} - C_{-1}$.

Lemma 2.5. *(i)* $\det\Lambda = \det K = (g+1)\det A = (g+1)p_1\cdots p_{g-1}C_{-1}$.
(ii) Let ν_Γ be a shift operator, $\nu_\Gamma:\ \mathbb{R}^g \to \mathbb{R}^g;\ (z_i)_{i=1,\cdots,g} \mapsto (z_i+C_{-1})_{i=1,\cdots,g}$. Then $J'(\Gamma) \simeq J(\Gamma)/\{P \sim \nu_\Gamma(P) \mid P \in J(\Gamma)\}$.

The proof is elementary and left for readers.

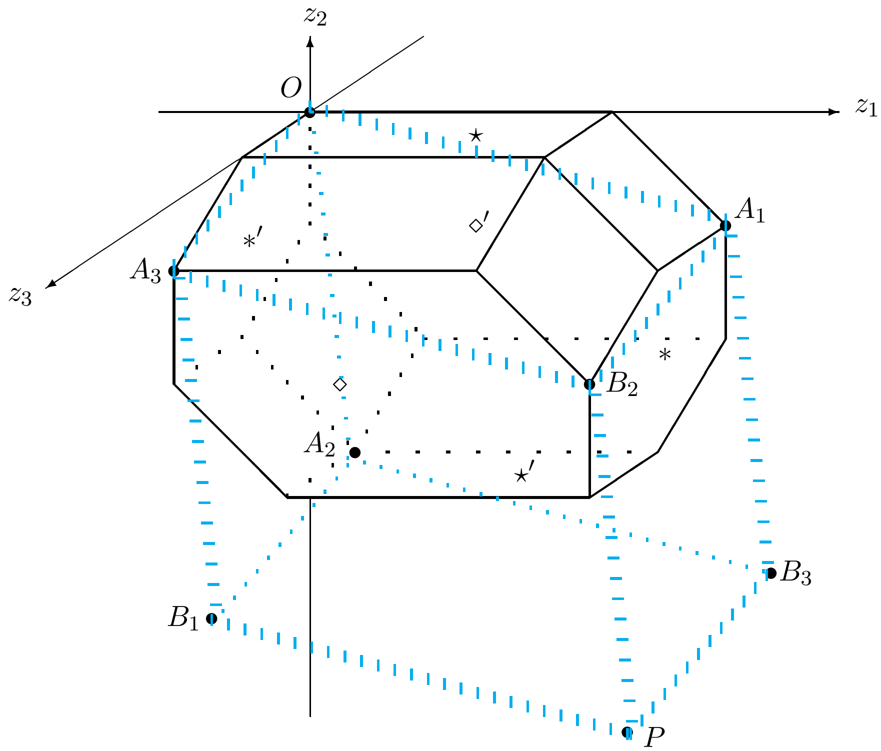

FIGURE 4. Image of η in $g = 3$

3. Isolevel set of ultra-discrete periodic Toda

3.1. **Periodic Toda lattice.** We review the known results on the $(g+1)$-periodic Toda lattice (1.1). We define a matrix $M^t(y)$ besides the Lax matrix $L^t(y)$ (1.2) on the phase space $\mathcal{U}$:

$$M^t(y) = \begin{pmatrix} I_2^t & 1 & & & \\ & I_3^t & 1 & & \\ & & \ddots & \ddots & \\ & & & I_{g+1}^t & 1 \\ y & & & & I_1^t \end{pmatrix}.$$

Proposition 3.1. [4] *(i) The system* (1.1) *is equivalent to the Lax form*

$$L^{t+1}(y)M^t(y) = M^t(y)L^t(y).$$

(ii) The system (1.1) *preserves the characteristic polynomial of* $L^t(y)$, $\det(x\,\mathbb{I} + L^t(y))$.

Proof. (i) Set $R^t(y)$ as

$$R^t(y) = \begin{pmatrix} 1 & & & (-1)^g \frac{V_1^t}{y} \\ V_2^t & 1 & & \\ & \ddots & \ddots & \\ & & V_{g+1}^t & 1 \end{pmatrix}.$$

The system (1.1) is equivalent to $R^{t+1}(y)M^{t+1}(y) = M^t(y)R^t(y)$. By the fact $L^t(y) = R^t(y)M^t(y)$, we have

$$L^{t+1}(y)M^t(y) = R^{t+1}(y)M^{t+1}(y)M^t(y) = M^t(y)R^t(y)M^t(y) = L^t(y)M^t(y).$$

(ii) From the Lax form we obtain $\det(x\,\mathbb{I}+L^{t+1}(y)) = \det(x\,\mathbb{I}+M^t(y)L^t(y)(M^t(y))^{-1}) = \det(x\,\mathbb{I}+L^t(y))$. $\square$

We define the (complex) spectral curve γ_c given by

$$\begin{aligned} f(x,y) =& y\det(\mathbb{I}\,x + L^t(y)) \\ =& y^2 + y(x^{g+1} + c_g x^g + \cdots + c_0) + c_{-1} = 0. \end{aligned} \tag{3.1}$$

Concretely, c_i is given by (for simplicity, we write $I_i^t = I_i, V_i^t = V_i$ and so on)

$$\begin{aligned} c_g &= \sum_{1\le i\le g+1} I_i + \sum_{1\le i\le g+1} V_i, \\ c_{g-1} &= \sum_{1\le i<j\le g+1} (I_iI_j) + \sum_{1\le i<j\le g+1} (V_iV_j) + \sum_{1\le i,j\le g+1, j\ne i,i-1} (I_iV_j), \\ &\vdots \\ c_0 &= \prod_{i=1}^{g+1} I_i + \prod_{i=1}^{g+1} V_i, \\ c_{-1} &= \prod_{i=1}^{g+1} I_iV_i. \end{aligned} \tag{3.2}$$

For generic c_i, γ_c is a hyperelliptic curve. Since (1.1) is invariant under $(I_i,V_i)_{1\le i\le g+1} \mapsto (I_ic_g, V_ic_g)_{1\le i\le g+1}$, we can set $c_g = 1$ without loss of generality.

Proposition 3.2. [5] *Under the condition $\prod_{k=1}^{g+1} V_k^t \ne \prod_{k=1}^{g+1} I_k^t$, the system (1.1) is equivalent to the system:*

$$\begin{aligned} I_i^{t+1} &= V_i^t + I_i^t \frac{1-\prod_{k=1}^{g+1}\frac{V_k^t}{I_k^t}}{1+\sum_{j=1}^{g}\prod_{k=1}^{j}\frac{V_{i-k}^t}{I_{i-k}^t}}, \\ V_i^{t+1} &= \frac{I_{i+1}^t V_i^t}{I_i^{t+1}}. \end{aligned} \tag{3.3}$$

3.2. **Ultra-discrete Toda lattice.** Suppose

$$V_i^t > 0,\ I_i^t > 0,$$
$$\prod_{i=1}^{g+1} V_i^t < \prod_{i=1}^{g+1} I_i^t. \tag{3.4}$$

In the UD-limit $\lim_{\varepsilon\to 0} \mathrm{Log}_\varepsilon$ with the scale transformation $I_i = \mathrm{e}^{-\frac{Q_i}{\varepsilon}}, V_i = \mathrm{e}^{-\frac{W_i}{\varepsilon}}$, the system (3.3) becomes the UD-pToda lattice (1.4). Simultaneously, the limit of the conserved quantities $c_i = \mathrm{e}^{-\frac{C_i}{\varepsilon}}$ become

$$\begin{aligned}
&C_g = \min[\min_{1\le i\le g+1} Q_i, \min_{1\le i\le g+1} W_i],\\
&C_{g-1} = \min[\min_{1\le i<j\le g+1}(Q_i+Q_j), \min_{1\le i<j\le g+1}(W_i+W_j), \min_{1\le i,j\le g+1, j\neq i,i-1}(Q_i+W_j)],\\
&\vdots\\
&C_0 = \min[\sum_{i=1}^{g+1} Q_i, \sum_{i=1}^{g+1} W_i],\\
&C_{-1} = \sum_{i=1}^{g+1}(Q_i+W_i),
\end{aligned} \tag{3.5}$$

which are preserved under (1.4) by construction. From the assumption (3.4), we have

$$\sum_{i=1}^{g+1} W_i^t > \sum_{i=1}^{g+1} Q_i^t.$$

We can set $C_g = 0$ without loss of generality corresponding to $c_g = 1$.

Next, we reconstruct the tropical curve $\tilde{\Gamma}_C$ by the UD-limit of the real part of the spectral curve γ_c. We write $\gamma_{\mathbb{R}}$ for the real part of $\gamma = \gamma_c$. Then the image of the map $\mathrm{Log}^2 :\ \mathbb{C}^2 \to \mathbb{R}^2;\ (x,y) \mapsto (\log|x|, \log|y|)$ of $\gamma_{\mathbb{R}}$ is the rim of the amoeba of γ.

In taking the UD-limit of the equation (3.1) with the scale transformation $c_i = e^{-\frac{C_i}{\varepsilon}}, |x| = e^{-\frac{X}{\varepsilon}}$ and $|y| = e^{-\frac{Y}{\varepsilon}}$, we have the following:

(i) $x > 0, y > 0$ leads to a contradiction.

(ii) $x < 0, y > 0$. We have

$$\Gamma_2 : \begin{cases}
\min[2Y, C_{-1}, (g+1)X+Y, (g-1)X+Y+C_{g-1}, \dots, Y+C_0] & \\
= \min[gX+Y+C_g, (g-2)X+Y+C_{g-2}, \dots, X+Y+C_1] & (g : \text{odd})\\
\min[2Y, C_{-1}, gX+Y+C_g, (g-2)X+Y+C_{g-2}, \dots, Y+C_0] & \\
= \min[(g+1)X+Y, (g-1)X+Y+C_{g-1}, \dots, X+Y+C_1] & (g : \text{even})
\end{cases}$$

(iii) $x < 0, y < 0$. We have

$$\Gamma_3 : \begin{cases}
\min[2Y, C_{-1}, gX+Y+C_g, (g-2)X+Y+C_{g-2}, \dots, X+Y+C_1] & \\
= \min[(g+1)X+Y, (g-1)X+Y+C_{g-1}, \dots, Y+C_0] & (g : \text{odd})\\
\min[2Y, C_{-1}, (g+1)X+Y, (g-1)X+Y+C_{g-1}, \dots, X+Y+C_1] & \\
= \min[gX+Y+C_g, (g-2)X+Y+C_{g-2}, \dots, Y+C_0] & (g : \text{even})
\end{cases}$$

(iv) $x > 0, y < 0$. We have

$$\Gamma_4 : \min[2Y, C_{-1}] = \min[(g+1)X + Y, gX + Y + C_g, \ldots, Y + C_0].$$

Then we obtain the following.

Proposition 3.3. *For generic C_i's which satisfy (2.1),*

$$\tilde{\Gamma}_C = \Gamma_2 \cup \Gamma_3 = \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

hold.

Fig. 5 shows $\gamma_{\mathbb{R}}$, Γ_2, Γ_3 and Γ_4 in the case of $g = 2$.

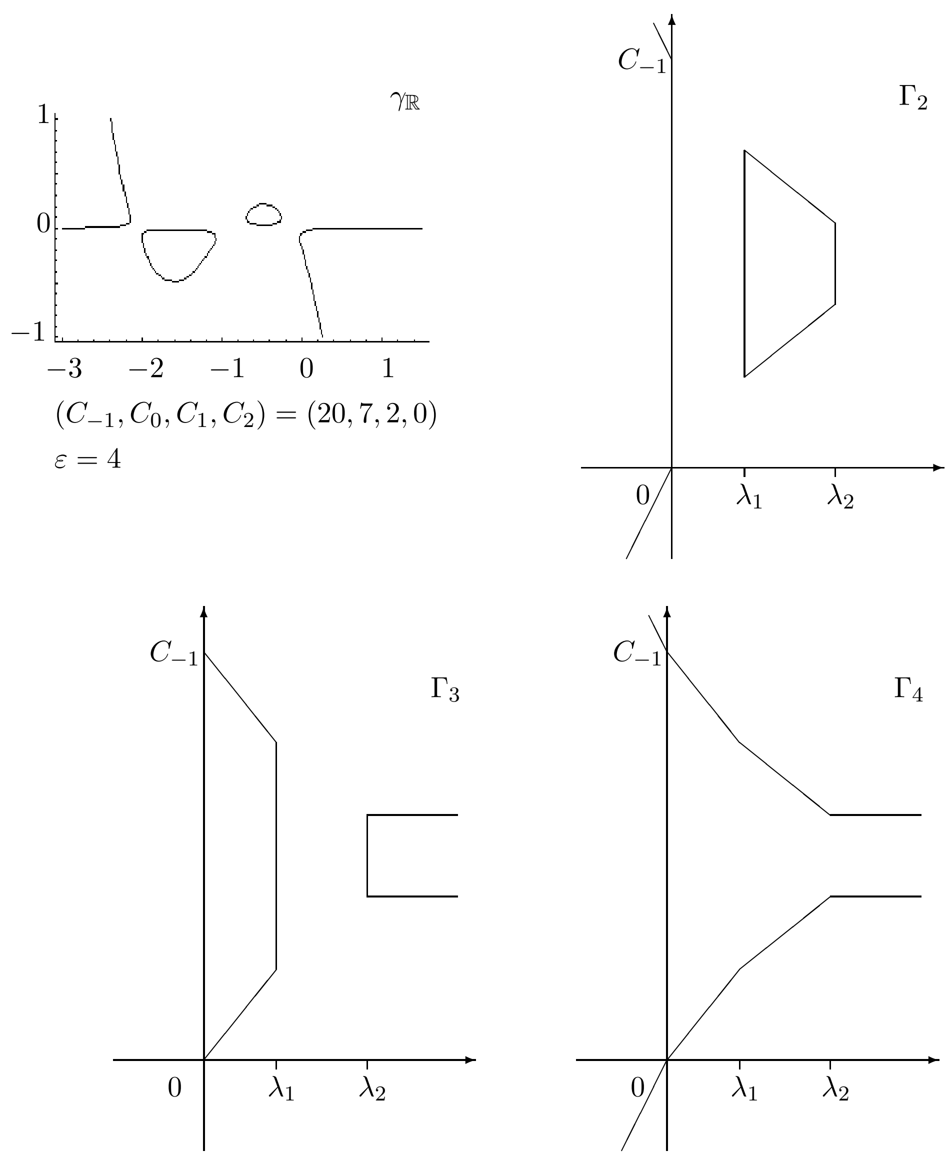

FIGURE 5. Real and tropical curves

3.3. **Eigenvector map.** Let $\mathcal{T}$ be the phase space of the ultra-discrete $(g+1)$-periodic Toda lattice, and $\mathcal{C}$ be the moduli space of the compact tropical curves Γ_C:

$$\mathcal{T} = \left\{(Q_1, \cdots, Q_{g+1}, W_1, \cdots, W_{g+1}) \in \mathbb{R}^{2g+2} \mid \sum_{i=1}^{g+1} Q_i < \sum_{i=1}^{g+1} W_i\right\},$$

$$\mathcal{C} = \{C = (C_{-1}, \cdots, C_{g-1}, C_g)\} \simeq \mathbb{R}^{g+2}.$$

We define a map $\Phi: \mathcal{T} \to \mathcal{C}$ by (3.5), and set $\mathcal{T}_C = \Phi^{-1}(C)$ for $C \in \mathcal{C}$.

Conjecture 3.4. *For a generic $C = (C_{-1}, \cdots, C_{g-1}, C_g = 0) \in \mathcal{C}$ which satisfies* (2.1), *following are satisfied:*
(i) $\mathcal{T}_C \simeq J(\Gamma_C)$.
(ii) *Suppose $C \in \mathbb{Z}^{g+2}$, and let $(\mathcal{T}_C)_\mathbb{Z}$ and $J_\mathbb{Z}(\Gamma_C)$ be the sets of lattice points in $\mathcal{T}_C$ and in $J(\Gamma_C)$ respectively. Then the isomorphism of* (i) *induces the bijection between $(\mathcal{T}_C)_\mathbb{Z}$ and $J_\mathbb{Z}(\Gamma_C)$. In particular, we have $|(\mathcal{T}_C)_\mathbb{Z}| = \det \Lambda$.*

Remark 3.5. This conjecture claims that we need only a compact part Γ_C of $\tilde{\Gamma}_C$ to construct the isolevel set $\mathcal{T}_C$.

In the rest of this section, we construct the isomorphism $\pi: \mathcal{T}_C \xrightarrow{\sim} J(\Gamma_C)$ in the case of $g = 1, 2$ and 3, by applying the technique of eigenvector map, which is essentially the same with Sklyanin's separation of variable in our case (for example see [2, 10]). The isomorphism π is a composition of isomorphisms:

$$\mathcal{T}_C \overset{\psi}{\to} \mathcal{D}^g(\Gamma_C) \overset{\eta}{\to} J(\Gamma_C),$$

where ψ is called the eigenvector map (or separation of variables) and η is the Abel-Jacobi map (2.3).

Remark 3.6. By concrete computation we also conjecture the following. Define a translation operator v as

$$v: J(\Gamma_C) \to J(\Gamma_C);\ z \mapsto z + (\lambda_1, \lambda_2 - \lambda_1, \ldots, \lambda_g - \lambda_{g-1}).$$

Then the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{T}_C & \xrightarrow{\pi} & J(\Gamma_C) \\ \downarrow T & & \downarrow v \\ \mathcal{T}_C & \xrightarrow{\pi} & J(\Gamma_C) \end{array}$$

i. e. the flow of the UD-pToda is linearized on the tropical Jacobian. It is easy to check it in the case of $g = 1$.

First we discuss the discrete case. Let us consider the eigenvector ϕ of the Lax matrix $L^t(y)$. Then ϕ is given by

$$\phi = {}^t(f_1, f_2, \ldots, f_g, -f_{g+1}),$$

where $f_i (i = 1, 2, \ldots, g)$ is

$$f_i = \det \begin{array}{c} \begin{matrix} 1 & 2 & \cdots & i & \cdots & g \end{matrix} \\ \begin{pmatrix} l_{11}+x & l_{12} & \cdots & l_{1,g+1} & \cdots & l_{1,g} \\ l_{21} & l_{22}+x & & l_{2,g+1} & & l_{2,g} \\ \vdots & \vdots & & \vdots & & \vdots \\ l_{g,1} & l_{g,2} & \cdots & l_{g,g+1} & \cdots & l_{g,g}+x \end{pmatrix} \end{array}$$

and

$$f_{g+1} = \det \begin{pmatrix} l_{11}+x & l_{12} & \cdots & l_{1,g} \\ l_{21} & l_{22}+x & \cdots & l_{2,g} \\ \vdots & \vdots & & \vdots \\ l_{g,1} & l_{g,2} & \cdots & l_{g,g}+x \end{pmatrix},$$

where $l_{ij} = (L^t(y))_{ij}$. The equation $f_{g+1}(x) = 0$ has the solution $x_1, x_2, \ldots, x_g$, each of which defines two points on γ_c: $(x_i, y_i), (x_i, y_i')$, where one of them (we assume that is (x_i, y_i)) leads $f_j = 0$ for all j. We choose $\{(x_i, y_i) \mid i = 1, 2, \ldots, g\}$ or $\{(x_i, y_i') \mid i = 1, 2, \ldots, g\}$ as a representative of $\mathrm{Pic}^g(\gamma_c)$. In the discrete case, this map induces an injection $\mathcal{U}_c \hookrightarrow \mathrm{Pic}^g(\gamma_c)$, and the evolution equation (1.1) is linearized on the Jacobi variety of γ_c, $\mathrm{Jac}(\gamma_c) \simeq \mathrm{Pic}^g(\gamma_c)$ (Cf. [1, 5, 10]).

3.4. **The case of $g = 1$.** The Lax matrix is

$$L^t(y) = \begin{pmatrix} a_1 & 1 - \frac{b_1}{y} \\ b_2 - y & a_2 \end{pmatrix}$$

and the conserved quantities are

$$c_{-1} = b_1 b_2, \ c_0 = a_1 a_2 - b_1 - b_2, \ c_1 = a_1 + a_2.$$

When $f_2 = a_1 + x = 0$, (3.1) becomes

$$f(x, y) = (y - b_1)(y - b_2) = 0.$$

Thus we define the map $\mathcal{U}_c \to \gamma_c$ by $u^t \mapsto (x_1 = -a_1, y_1 = b_1)$.

In the ultra-discrete limit, the map $\psi : \mathcal{T}_C \to \Gamma_C$ is given by

$$(Q_1, Q_2, W_1, W_2) \mapsto (X_1 = \min[Q_2, W_1], Y_1 = Q_1 + W_1) \in \Gamma_2$$

where $C_{-1} = Q_1 + Q_2 + W_1 + W_2$, $C_0 = Q_1 + Q_2$ and $C_1 = \min[Q_1, Q_2, W_1, W_2] = 0$. We see the following:

Proposition 3.7. *The map ψ is bijective. In particular, $\mathcal{T}_C \simeq J(\Gamma_C)$.*

Proof. By construction it is obvious that the image of ψ is included in Γ_C. Inversely, solving

$$a_1 = -x,\ \ b_1 = y$$
$$a_2 = \frac{c_0 + b_1 + b_2}{a_1},\ \ b_2 = \frac{c_{-1}}{y}$$

for I_i, V_j, we have the solutions (I_i, V_j) and (I_i', V_j') $(i, j = 1, 2)$:

$$I_1 + I_1' = -\frac{c_0 + 2y}{x}, \quad I_2 + I_2' = -\frac{x(2c_{-1} + c_0 y)}{c_{-1} + c_0 y + y^2}$$
$$V_1 = \frac{y}{I_1},\ V_1' = \frac{y}{I_1'},\ V_2 = \frac{c_{-1}}{yI_2},\ V_2' = \frac{c_{-1}}{yI_2'},$$

where we assume $I_i \geq I_i'$. Only (I_i, V_j) satisfies the assumption (3.4). By the UD-limit, we have the inverse of ψ as

$$\begin{aligned} Q_1 =& \min[C_0, Y] - X \\ Q_2 =& X + \min[C_{-1}, C_0 + Y] - \min[C_{-1}, C_0 + Y, 2Y] \\ W_1 =& Y - Q_1 \\ W_2 =& C_{-1} - Y - Q_2. \end{aligned}$$

□

3.5. **The case of $g = 2$.** In this and the next subsection we denote min[ ] simply by [ ]. The Lax matrix is

$$L^t(y) = \begin{pmatrix} a_1 & 1 & \frac{b_1}{y} \\ b_2 & a_2 & 1 \\ y & b_3 & a_3 \end{pmatrix},$$

and the conserved quantities are

$$c_{-1} = b_1 b_2 b_3,\ c_0 = a_1 a_2 a_3 - a_2 b_1 - a_3 b_2 - a_1 b_3,$$
$$c_1 = a_1 a_2 + a_2 a_3 + a_3 a_1 - b_1 - b_2 - b_3,\ c_2 = a_1 + a_2 + a_3.$$

The UD-pToda (1.4) is

$$\begin{array}{rcl} Q_i^{t+1} & = & [W_i^t, Q_i^t - X_i^t] \\ W_i^{t+1} & = & Q_{i+1}^t + W_i^t - Q_i^{t+1} \end{array} \tag{3.6}$$

with

$$X_i^t = [0, W_{i-1}^t - Q_{i-1}^t, W_{i-1}^t + W_{i-2}^t - Q_{i-1}^t - Q_{i-2}^t],$$

and the conserved quantities (3.5) become

$$
\begin{aligned}
C_2 =&[Q_1, Q_2, Q_3, W_1, W_2, W_3] = 0\\
C_1 =&[Q_1 + Q_2, Q_2 + Q_3, Q_3 + Q_1, W_1 + W_2, W_2 + W_3, W_3 + W_1,\\
&\quad Q_1 + W_2, Q_2 + W_3, Q_3 + W_1]\\
C_0 =&[Q_1 + Q_2 + Q_3, W_1 + W_2 + W_3]\\
C_{-1} =&Q_1 + Q_2 + Q_3 + W_1 + W_2 + W_3.
\end{aligned}
$$

The tropical spectral curve is the set sum of

$$\Gamma_2 : [2Y, C_{-1}, 2X + Y + C_2, Y + C_0] = [3X + Y, X + Y + C_1]$$

and

$$\Gamma_3 : [2Y, C_{-1}, 3X + Y, X + Y + C_1] = [2X + Y + C_2, Y + C_0].$$

The eigenvector of the Lax matrix is given by

$$f_1 = \begin{vmatrix} \frac{b_1}{y} & 1 \\ 1 & a_2 + x \end{vmatrix},\ f_2 = \begin{vmatrix} a_1 + x & \frac{b_1}{y} \\ b_2 & 1 \end{vmatrix},\ f_3 = \begin{vmatrix} a_1 + x & 1 \\ b_2 & a_2 + x \end{vmatrix}.$$

When $f_3 = 0$, (3.1) reduces to

$$f(x, y) = (y - b_1(x + a_2))(y - b_3(x + a_1)) = 0.$$

The solutions are

$$
\begin{aligned}
&x_1 + x_2 = -a_1 - a_2,\ x_1x_2 = a_1a_2 - b_2\\
&y_i = b_1(x_i + a_2),\ y_i' = b_3(x_i + a_1) \quad \text{for } i = 1, 2.
\end{aligned}
\tag{3.7}
$$

For the UD-limit we use another expression of y_i:

$$y_i = \frac{c_{-1}}{b_3(x_i + a_1)}.$$

Under the assumption $x_1, x_2 < 0$, $y_1 < 0$, $y_2 > 0$ (for small $\varepsilon > 0$), the UD-limit of (3.7) leads:

$$
\begin{aligned}
X_1 =&[Q_2, Q_3, W_1, W_2]\\
X_2 =&[Q_2 + Q_3, W_1 + W_2, Q_3 + W_1] - X_1
\end{aligned}
\tag{3.8}
$$

and

$$
Y_1 = \begin{cases} Y_1^a := Q_1 + W_1 + X_1 & \text{if } X_1 < [Q_3, W_2] \\ Y_1^b := C_{-1} - (Q_3 + W_3 + X_1) & \text{if } X_1 < [Q_2, W_1] \end{cases}
$$

$$
Y_2 = \begin{cases} Y_2^a := Q_1 + W_1 + [Q_3, W_2] & \text{if } X_2 > [Q_3, W_2] \\ Y_2^b := C_{-1} - (Q_3 + W_3 + [Q_2, W_1]) & \text{if } X_2 > [Q_2, W_1] \end{cases}.
$$

The following lemma can be proved elementarily.

Lemma 3.8. (i) $C_2(=0) \le X_1 \le C_2 + \lambda_1 \le X_2 \le C_2 + \lambda_2$.
(ii) $X_1 = [[Q_2, W_1], [Q_3, W_2]],\ X_2 \ge \max[[Q_2, W_1], [Q_3, W_2]]$.
(iii) *If* $[Q_2, W_1] = [Q_3, W_2]$, *then*
(iii-1)$X_1 = X_2$ *and thus* $Y_1^a = Y_2^a$ *and* $Y_1^b = Y_2^b$
or (iii-2) $Y_1^a = Y_1^b$ *and* $Y_2^a = Y_2^b$
hold.

By Lemma 3.8, the correspondence between $(Q_1, Q_2, Q_3, W_1, W_2, W_3) \in \mathcal{T}_C$ and $(X_1, Y_1) + (X_2, Y_2) \in \mathrm{Div}^2_{\mathrm{eff}}(\Gamma_C)$ is uniquely expanded as a continuous map $\psi : \mathcal{T}_C \to \mathrm{Div}^2_{\mathrm{eff}}(\Gamma_C)$ by (3.8) and

$$\left.\begin{aligned} Y_1 &= Y_1^a = Q_1 + W_1 + [Q_2, W_1] \\ Y_2 &= Y_2^b = C_{-1} - (Q_3 + W_3 + [Q_2, W_1]) \end{aligned}\right\} \text{ if } [Q_2, W_1] \le [Q_3, W_2],$$
$$\left.\begin{aligned} Y_1 &= Y_1^b = C_{-1} - (Q_3 + W_3 + [Q_3, W_2]) \\ Y_2 &= Y_2^a = Q_1 + W_1 + [Q_3, W_2] \end{aligned}\right\} \text{ if } [Q_3, W_2] \le [Q_2, W_1].$$

(When $X_1 = X_2$, we can exchange Y_1 and Y_2.)

Lemma 3.9. *The image of* ψ *is included in* $\mathcal{D}^2(\Gamma_C)$, *i.e. if* $X_1 = X_2$, *then* (X_1, Y_1) *or* (X_2, Y_2) *is at the end point of* α_{12}.

Proof. By Lemma 3.8(ii), we have $[Q_2, W_1] = [Q_3, W_2]$. Without loss of generality we can assume $Q_1 = 0$. (i) $Q_2 = Q_3 \le W_1, W_2$ leads $C_1 = Q_2$ and $C_2 = 2Q_2$, which contradict to the smoothness (2.1). (ii) $Q_2 = W_2 < Q_3, W_1$ leads $X_1 = Q_2$ and $X_2 > Q_2$; which is a contradiction. (iii) $W_1 = Q_3 \le Q_2, W_2$ leads $C_1 = W_1$ and $Y_1^a = 2C_1$. (iv) $W_1 = W_2 \le Q_2, Q_3$ leads $C_1 = W_1$ and $Y_1^a = 2C_1$. $\square$

Inversely, solving

$$\begin{aligned} a_1 &= -\frac{x_1 y_1 - x_2 y_2}{y_1 - y_2},\ b_1 = \frac{y_1 - y_2}{x_1 - x_2} \\ a_2 &= \frac{x_1 y_2 - x_2 y_1}{y_1 - y_2},\ b_2 = -\frac{y_1 y_2 (x_1 - x_2)^2}{(y_1 - y_2)^2} \\ a_3 &= \frac{c_0 a_1 b_3 + a_2 b_1}{a_1 a_2 - b_2},\ b_3 = -\frac{c_{-1}(y_1 - y_2)}{y_1 y_2 (x_1 - x_2)} \end{aligned}$$

for I_i, V_j, we have (e.g.)

$$I_1 + I_1' = \frac{c_0(x_1 - x_2) + 2(x_1 y_2 - x_2 y_1)}{x_1 x_2 (x_1 - x_2)}.$$

By the UD-limit, we have the inverse of ψ if $X_1 < X_2$

$$
\begin{aligned}
Q_1 =&[C_0 + X_1, U_2] - (2X_1 + X_2) \\
Q_2 =&2X_1 + [C_{-1} + U_1, Y_1 + Y_2 + U_2, C_0 + [X_1 + Y_1 + Y_2, X_2 + 2[Y_1, Y_2]]] \\
&- [Y_1, Y_2] - [C_{-1} + 2X_1, C_0 + X_1 + U_2, 2U_2] \\
Q_3 =&X_1 + X_2 + [Y_1, Y_2] + [C_{-1} + U_1, C_0 + X_1 + Y_1 + Y_2] \\
&- [C_{-1} + 2U_1, C_0 + X_1 + Y_1 + Y_2 + U_1, 2X_1 + 2Y_1 + 2Y_2] \\
W_1 =& [Y_1, Y_2] - X_1 - Q_1 \\
W_2 =& Y_1 + Y_2 + 2X_1 - 2[Y_1, Y_2] - Q_2 \\
W_3 =& [C_{-1} + [Y_1, Y_2] - Y_1 - Y_2 - X_1 - Q_3
\end{aligned}
$$

with

$$U_1 = [X_1 + Y_1, X_2 + Y_2], \qquad U_2 = [X_1 + Y_2, X_2 + Y_1].$$

By Lemma 3.9, the inverse is uniquely expanded as a continuous map to the case of $X_1 = X_2$.

Now we have the following.

Proposition 3.10. *The UD-eigenvector map $\psi : \mathcal{T}_C \to \mathcal{D}^2(\Gamma_C)$ is bijective.*

3.6. **The case of $g = 3$.** In the case of $g = 3$ we present the ultra-discrete eigenvector map $\psi : \mathcal{T}_C \to \mathcal{D}^3(\Gamma_C)$. However, for the reason of complexity, we will omit to present the inverse mapping and to prove the bijectivity.

The solutions of $f_4 = 0$ and $f(x, y) = 0$ are

$$
\begin{aligned}
&x_1 + x_2 + x_3 = -a_1 - a_2 - a_2 \\
&x_1x_2 + x_2x_3 + x_3x_1 = a_1a_2 + a_2a_3 + a_3a_1 - b_2 - b_3 \\
&x_1x_2x_3 = -a_1a_2a_3 + a_1b_3 + a_3b_2 \\
&y_i = b_1((a_2 + x_i)(a_3 + x_i) - b_3)), \ y'_i = b_3((a_1 + x_i)(a_2 + x_i) - b_2)) \ \text{ for } i = 1, 2, 3.
\end{aligned}
\tag{3.9}
$$

For the UD-limit we use other expressions of y_i:

$$y_i = b_1b_2\frac{a_3 + x_i}{a_1 + x_i} = \frac{c_{-1}}{b_4((a_1 + x_i)(a_2 + x_i) - b_2))}.$$

The UD-limit of (3.9) leads the UD-eigenvector map $\psi : \mathcal{T}_C \to \mathcal{D}^3(\Gamma_C)$:

$$
\begin{aligned}
X_1 =&[Q_2, Q_3, Q_4, W_1, W_2, W_3] \\
X_2 =&[Q_2 + Q_3, Q_3 + Q_4, Q_2 + Q_4, W_1 + W_2, W_2 + W_3, W_1 + W_3, Q_4 + W_1, \\
&Q_4 + W_2, Q_2 + W_3, Q_3 + W_1] - X_1 \\
X_3 =&[Q_2 + Q_3 + Q_4, W_1 + Q_3 + Q_4, W_1 + W_2 + Q_4, W_1 + W_2 + W_3] - (X_1 + X_2) \\
Y_i =&Y_i^{s_i} \ \text{ for } i = 1, 2, 3,
\end{aligned}
$$

where

$$\begin{aligned}
Y_i^1 =&Q_1+W_1+[2X_i,X_i+[Q_3,Q_4,W_2,W_3],[Q_3+Q_4,W_2+W_3,Q_4+W_2]]\\
Y_i^2 =&Q_1+W_1+Q_2+W_2+[Q_4,W_3,X_i]-[Q_2,W_1,X_i]\\
Y_i^3 =&C_{-1}-(Q_4+W_4+[2X_i,X_i+[Q_2,Q_3,W_1,W_2],[Q_2+Q_3,W_1+W_2,Q_3+W_1]])
\end{aligned}$$

and s_i is defined as follows.
(i) Set A^1,A^2,A^3,B^1,B^3 as $A^1=[Q_2,W_1], A^2=[Q_3,W_2], A^3=[Q_4,W_3]$,
$B^1=[Q_3+Q_4,W_2+W_3,Q_4+W_2], B^3=[Q_2+Q_3,W_1+W_2,Q_3+W_1]$, and define s_1 by
$s_1=1$ if $A^1\le[A^2,A^3]$,
$s_1=2$ if $A^2\le[A^3,A^1]$,
$s_1=3$ if $A^3\le[A^1,A^2]$.
If s_1 has two or more possibilities, choose one of them.
(ii) Define s_2 and s_3 so that $s_i\neq s_j (i,j=1,2,3)$ by
$s_2=1$ if $X_2+[A^2,A^3]<B^1$,
$s_2=2$ if $A^1<X_2<A^3$ or $A^3<X_2<A^1$,
$s_2=3$ if $X_2+[A^1,A^2]<B^3$
and
$s_3=1$ if $X_3+[A^2,A^3]>B^1$,
$s_3=2$ if $X_3>\max[A^1,A^3]$,
$s_3=3$ if $X_3+[A^1,A^2]>B^3$.
(iii) If both s_2 and s_3 are not determined by (ii), then choose s_2 and s_3 arbitrarily under keeping $s_i\neq s_j (i,j=1,2,3)$.

4. From the UD-pToda to the pBBS

4.1. The structure of $\mathcal{T}_C$. Fix a generic $C\in\mathcal{C}$ with $C_g=0$. Define a shift operator $s:\ \mathcal{T}_C\to\mathcal{T}_C$;

$$(Q_1,\cdots,Q_{g+1},W_1,\cdots,W_{g+1})\mapsto(Q_2,\cdots,Q_{g+1},Q_1,W_2,\cdots,W_{g+1},W_1). \tag{4.1}$$

Note $s^{g+1}=id$. We define a subspace T_C^0 of $\mathcal{T}_C$:

(4.2)
$$T_C^0=\big\{(Q_1,\cdots,Q_{g+1},W_1,\cdots,W_{g+1})\in\mathcal{T}_C \mid \text{(a) } W_1>0, \text{ and (b) } Q_1=0 \text{ or } W_{g+1}=0.\big\}.$$

We write T_C^i for the set given by

$$T_C^i=\{s^i(\tau)\mid\tau\in T_C^0\},\quad\text{for } i\in\mathbb{Z}.$$

Proposition 4.1. (i) $T_C^i\cap T_C^j=\emptyset$ *for* $i\neq j \mod g+1$, (ii) $\mathcal{T}_C=\bigcup_{i=0}^{g}T_C^i$.

First we show

Lemma 4.2. *If $\tau = (Q_1, \cdots, Q_{g+1}, W_1, \cdots, W_{g+1}) \in T_C^0$, then $Q_i > 0$ for $2 \le i \le g$, and $W_j > 0$ for $1 \le j \le g$.*

Proof. Recall that the conserved quantity C_{g-1} (3.2) for $\mathcal{T}_C$ satisfies $C_{g-1} > 0$. For $\tau = (Q_1, \cdots, Q_{g+1}, W_1, \cdots, W_{g+1}) \in \mathcal{T}_C$, the following properties (b1) and (b2) hold:
(b1) When $Q_1 = 0$, we have

$$C_{g-1} = \min[\min_{2 \le i \le g+1} Q_i, \min_{2 \le i \le g} W_i, W_1 + W_{g+1}] > 0. \tag{4.3}$$

Thus we obtain $Q_i > 0$ for $2 \le i \le g+1$ and $W_j > 0$ for $2 \le i \le g$.
(b2) When $W_{g+1} = 0$, we have

$$C_{g-1} = \min[\min_{2 \le i \le g} Q_i, \min_{1 \le i \le g} W_i, Q_1 + Q_{g+1}] > 0. \tag{4.4}$$

Thus we obtain $Q_i > 0$ for $2 \le i \le g$ and $W_j > 0$ for $1 \le i \le g$.

If we further assume $\tau \in T_C^0$, we have $W_1 > 0$, and (b1) or (b2) is satisfied. Thus one obtains the claim. □

Proof. (Proposition 4.1)
(i) Note that

$$T_C^i = \{(Q_1, \cdots, Q_{g+1}, W_1, \cdots, W_{g+1}) \mid \text{(a) } W_{i+1} > 0, \text{ and (b) } Q_{i+1} = 0 \text{ or } W_i = 0\}.$$

We check that if $\tau = (Q_1, \cdots, Q_{g+1}, W_1, \cdots, W_{g+1}) \in T_C^0$ then it satisfies (a') $W_{i+1} = 0$, or (b') $Q_{i+1} > 0$ and $W_i > 0$, for $i = 1, \cdots, g$. For $i = 1, \cdots, g-1$, (b') is satisfied due to Lemma 4.2. For $i = g$, (b') is satisfied when $Q_1 = 0$ and (a') is satisfied when $W_{g+1} = 0$.
(ii) Is is trivial that $\mathcal{T}_C \supset \bigcup_{i=0}^{g} T_C^i$. We show $\mathcal{T}_C \subset \bigcup_{i=0}^{g} T_C^i$. Since $C_g = 0$, for $\tau \in \mathcal{T}_C$ we assume $Q_1 = 0$ or $W_{g+1} = 0$ without loss of the generality. When $Q_1 = 0$, (4.3) denotes $Q_2, \cdots, Q_{g+1}, W_2, \cdots, W_g > 0$ and $W_1 + W_{g+1} > 0$. Thus we see $\tau \in T_C^1$ when $W_1 = 0$, and $\tau \in T_C^0$ when $W_1 > 0$. In the same way, when $W_{g+1} = 0$ it is easy to see that (4.4) indicates $\tau \in T_C^0$. □

4.2. **Periodic BBS.** Fix $L \in \mathbb{Z}_{>0}$. The L-periodic box-ball system is a cellular automaton that the finite number of balls move in a periodic array of L boxes, where each box has one ball at most [16]. We assume that the number of balls $|\lambda|$ satisfies $2|\lambda| < L$. The time evolution of the pBBS from the time step t to $t+1$ is given as follows:

(1) Choose one ball and move it to the leftmost empty box to its right.
(2) Choose one of unmoved balls and move it as (1), ignoring the boxes to which and from which the balls were moved in this time step.
(3) Continue (2) until every ball moves once.

This system has conserved quantities parametrized by a non-decreasing array $\lambda = (\lambda_1, \cdots, \lambda_g) \in (\mathbb{Z}_{>0})^g$ such that $\sum_{i=1}^{g} \lambda_i = |\lambda|$ for some $g \in \mathbb{Z}_{>0}$. In the following we write 0 and 1 for "an empty box" and "an occupied box" respectively, and let $B_L \simeq \{0,1\}^{\times L}$

be the phase space of L-periodic BBS. We show examples of the evolution of $b(t) \in B_L$ as time t goes:

Example 4.3. The case of (i) $(L, \lambda_1) = (8, 3)$ and (ii) $(L, \lambda_1, \lambda_2) = (7, 1, 2)$:

(i)

t	$b(t)$
0	00111000
1	00000111
2	11100000
3	00011100
4	10000011
5	01110000

(ii)

t	$b(t)$	$\beta(b(t))$	$T^t(\beta(b(0)))$
0	0100110	$(0, 1, 2, 1, 2, 1)$	$(0, 1, 2, 1, 2, 1)$
1	1010001	$(1, 1, 1, 1, 3, 0)$	$(1, 1, 1, 1, 3, 0)$
2	0101100	$(0, 1, 2, 1, 1, 2)$	$(1, 2, 0, 1, 2, 1)$
3	0010011	$(0, 1, 2, 2, 2, 0)$	$(1, 2, 0, 2, 0, 2)$
4	1101000	$(2, 1, 0, 1, 3, 0)$	$(1, 0, 2, 3, 0, 1)$
5	0010110	$(0, 1, 2, 2, 1, 1)$	$(2, 0, 1, 1, 2, 1)$

Roughly speaking, g is the number of consecutive clusters of 1's, and $(\lambda_1, \cdots, \lambda_g)$ corresponds to the number of 1's in each cluster.

The injection from B_L to $\mathcal{T}$ is introduced in [5]. Fix a generic $C \in \mathcal{C} \cap \mathbb{Z}^{g+2}$ which satisfies (2.1) with $C_{-1} = L$ and $C_g = 0$, and set $\lambda = (\lambda_1, \cdots, \lambda_g)$ (2.2). Note that the generic condition for C corresponds to the condition: $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_g$. Let $B_{L,\lambda} \subset B_L$ be a set of the states whose conserved quantity is λ. Then the injection $\beta : B_{L,\lambda} \hookrightarrow (\mathcal{T}_C)_{\mathbb{Z}};\ b \mapsto (Q_1, \cdots, Q_{g+1}, W_1, \cdots, W_{g+1})$ is defined as follows:

(1) if the leftmost entry of b is 1, then set $Q_1 = \sharp$(the first consecutive 1's from the left), otherwise set $Q_1 = 0$.

(2) Set $W_i = \sharp$(the i-th consecutive 0's from the left) for $i = 1, \cdots, g+1$. If $Q_1 \neq 0$, set $Q_i = \sharp$(the i-th consecutive 1's from the left), otherwise set $Q_i = \sharp$(the $(i-1)$-th consecutive 1's from the left) for $i = 2, \cdots, g+1$.

Proposition 4.4. *$\beta : B_{L,\lambda} \to (T_C^0)_{\mathbb{Z}} := T_C^0 \cap \mathbb{Z}^{2(g+1)}$ is a bijection. In particular, we have the bijection between $(\mathcal{T}_C)_{\mathbb{Z}}/\{\tau \sim s(\tau) \mid \tau \in (\mathcal{T}_C)_{\mathbb{Z}}\}$ and $B_{L,\lambda}$, which leads to $|(\mathcal{T}_C)_{\mathbb{Z}}| = (g+1)|B_{L,\lambda}|$.*

Proof. By the definition of the map β, it is obvious $\beta(B_{L,\lambda}) \subset (T_C^0)_{\mathbb{Z}}$. From Lemma 4.2, each $\tau \in T_C^0$ gives the array $(Q_1, W_1, Q_2, \cdots, W_g, Q_{g+1}, W_{g+1})$ where $W_1, Q_2, \cdots, W_g > 0$. We define a map $\rho : (T_C^0)_{\mathbb{Z}} \to B_{L,\lambda}$ as follows: when $Q_1 = 0$, set $\rho(\tau)$ as

$$\underbrace{0 \cdots 0}_{W_1}\underbrace{1 \cdots 1}_{Q_2} \cdots \underbrace{1 \cdots 1}_{Q_{g+1}}\underbrace{0 \cdots 0}_{W_{g+1}}$$

where W_{g+1} can be zero. When $W_{g+1} = 0$, set $\eta(\tau)$ as

$$\underbrace{1 \cdots 1}_{Q_1}\underbrace{0 \cdots 0}_{W_1} \cdots \underbrace{0 \cdots 0}_{W_g}\underbrace{1 \cdots 1}_{Q_{g+1}}$$

where one of Q_1 and Q_{g+1} can be zero. In both cases, it is clear that $\beta \cdot \rho(\tau) = \tau$. Thus $\rho = \beta^{-1}$. ∎

From Prop. 4.1 and 4.4 we can put back $b(t)$ from the solution of the UD-pToda lattice with the initial state $\beta(b(0))$ (see Example 4.3 (ii)).

Lemma 2.5 clarifies the algebro-geometrical meaning of $J'(\Gamma_C)$ (2.4) which was first introduced in the study of the pBBS by Kuniba et al:

Theorem 4.5. [6, Theorem 3.11] *Let $J'_{\mathbb{Z}}(\Gamma_C)$ be the set of lattice points in $J'(\Gamma_C)$. Then the bijection between $B_{L,\lambda}$ and $J'_{\mathbb{Z}}(\Gamma_C)$ is induced by Kerov-Kirillov-Reshetikhin bijection.*

In the following diagram we summarize the relation among the UD-pToda, the pBBS and the tropical Jacobian:

$$\begin{array}{ccccc} B_{L,\lambda} & \overset{\beta}{\underset{/s}{\rightleftarrows}} & (\mathcal{T}_C)_{\mathbb{Z}} & \subset & \mathcal{T}_C \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ J'_{\mathbb{Z}}(\Gamma_C) & \leftarrow_{/\nu_\Gamma} & J_{\mathbb{Z}}(\Gamma_C) & \subset & J(\Gamma_C) \end{array} \tag{4.5}$$

Here $/s$ and $/\nu_\Gamma$ are the quotient maps respectively induced by the shift operators s (4.1) and ν_Γ at Lemma 2.5 (ii). The isomorphism of the right two downward maps are the claim in Conjecture 3.4. The diagram also indicates $J'(\Gamma_C) \simeq J(\Gamma_C)/\{P \sim s^*(P) \mid P \in J(\Gamma_C)\}$.

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